Group of Homotopy Spheres: Lightning Talk

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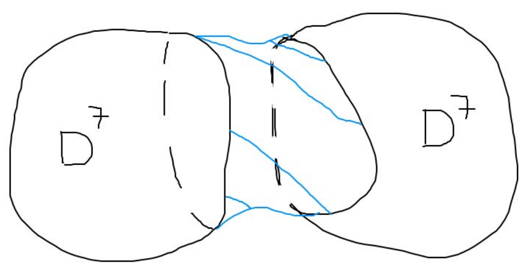
Throughout $n \geq 5$.

Theorem (Milnor, 1956)**.** *There exists a smooth manifold homeomorphic to the 7 sphere but not diffeomorphic.*



The idea was to glue two 7 discs along their boundary by a diffeomorphism. This opened the question to how many smooth structures can the topological sphere have. In [1962] Smale proved the h-cobordism theorem (at least in high dimensions) which implies that the gluing construction Milnor used is sufficient to construct *all* smooth structures on discs. It also allows you to consider homotopy spheres up to h-cobordism. KM leveraged this to prove:

Theorem (Kervaire-Milnor, 1963)**.** *The set of smooth manifolds that are homeomorphic to a sphere forms a group under connected sum that is finite and abelian. Call this group Θ_n.*

Thus the set of differentiable structures is not too bad. KM first show that there is a ses

$$0 \to bP_{n+1} \to \Theta_n \to \text{coker}(J) = \pi_n^s/\text{Im}(J)$$

Serre showed [1951] previously that the stable homotopy groups are finite. Then KM gave the following order of the finite cyclic groups (which they also prove)

$$|bP_{n+1}| = \begin{cases} 1, & n \equiv 0, 2 \bmod 4 \\ 1 \text{ or } 2, & n \equiv 1 \bmod 4 \\ 2^{2m-2}(2^{2m-1}-1)\text{numerator}\left(\frac{4B_m}{m}\right), & n \equiv 3 \bmod 4,\, n+1 = 4m \end{cases}$$

Where B_k is the (topologists) Bernoulli number. We can apply Lagrange theorem to get a relation between the group of smooth structures, the stable homotopy groups of spheres and the Bernoulli numbers!

Remark. They also announced some more exact formulas however these were not proven in their papers.